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On the systematic approach to the classification of differential equations by group theoretical methods

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ABSTRACT

Complete symmetry groups enable one to characterise fully a given differential equation. By considering the reversal of an approach based upon complete symmetry groups we construct new classes of differential equations which have the equations of Bateman, Monge–Ampère and Born–Infeld as special cases. We develop a symbolic algorithm to decrease the complexity of the calculations involved.

1. Introduction

Differential equations possess, in theory, an infinite number of symmetries.[3] However, in real applications, differential equations exhibit a quite finite number of useful symmetries. Independently of the possible integrability or nonintegrability of a given system of differential equations, symmetries can play a vital role towards the reduction of their order, determination of first integrals, generation of new solutions from known ones (even with trivial solutions as a starting point) and various other methods for obtaining solutions in closed form.

Initially Lie used group transformations for which the coefficient functions of the generator are functions of the independent and dependent variables only (point symmetries). Later the notion of symmetry was extended to contact transformations, *i.e.* group transformations in the jet-1 space with the extra requirement that the coefficient functions of the first extension of the generator be independent of the second derivative of each dependent variable. By the time Noether presented her celebrated Theorem in 1918 she was able to use generalised symmetries in which the coefficient functions of the symmetry can contain such derivatives of the dependent variables as makes sense.[4] In the last two decades nonlocal symmetries, *ie* symmetries with coefficient functions containing integrals which cannot be evaluated without a knowledge of the solution of the equation, have been introduced. The more general forms for the coefficient functions of the infinitesimal generator have been found to be useful in the resolution of many problems. Consequently the infinite number of such symmetries possessed by any differential equation may contain a subset of symmetries useful to the purpose at hand. Indeed in the case of partial differential equations an infinite sequence of nontrivial generalised symmetries reflects the integrability of the given equation.

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3 Without specifying the variable dependence in the coefficient functions of the infinitesimal generator one obtains a single determining equation which is underdetermined. The infinity follows.

4 For an ordinary differential equation the orders of the derivatives in the coefficient functions must be less than the order of the equation.

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Every group of symmetries is specific to a class of differential equations (the case of algebras is still not adequately treated in the sense that there exist different representations of the same group). In this way one is able to determine all differential equations which remain invariant under the action of a particular group of symmetries. The latter was termed a realisation of a complete symmetry group in [14] in 1994. The vehicle he used was the Kepler Problem for which in fact a fewer number of symmetries than given by Krause are needed to specify it completely [17]. Subsequently there has been a wide usage of this concept by various authors [15,3,4,16,20,5,6] with a basic theoretical approach performed in [1,2].

In actual fact this precise problem was communicated as early as 1974 in [8]. Bluman and Cole, in the last Section of their out-of-print textbook, discuss the heat equation,

$$u_{xx} - u_t = 0, \tag{1}$$

provide its Lie algebra with operators

$$\begin{aligned} &H_1 = \partial_x \qquad H_4 = x\partial_x + 2t\partial_t \\ &H_2 = \partial_t \qquad H_5 = xt\partial_x + t^2\partial_t - \left(\frac{1}{4}x^2 + \frac{1}{2}t\right)u\partial_u \\ &H_3 = u\partial_u \qquad H_6 = t\partial_x - \frac{1}{2}xu\partial_u \end{aligned} \tag{2}$$

and 'consider the problem of finding the most general partial differential equation invariant under the given multiparameter Lie group' restricting their investigation to second-order partial differential equations with two independent variables. They start from

$$u_{xx} = F(t, x, u, u_t, u_x, u_{tx}, u_{tt}) \tag{3}$$

and apply the suitable (in this case, second) extensions of the symmetries until they recover the equation under investigation (here, the heat equation).

From all the possible subalgebras for (1) they concentrate on the chain

$$\mathrm{span}\{H_1\} \subset \mathrm{span}\{H_1, H_2\} \subset \mathrm{span}\{H_1, H_2, H_3\} \subset \mathrm{span}\{H_2, H_6\} \subset \mathrm{span}\{H_2, H_4, H_6\} \subset \mathrm{span}\{H_1, H_2, H_5\}$$

and produce the most general differential equation for each subalgebra in turn until the ultimate to obtain

$$u_{xx} = u_t + \frac{u_x^2}{u}\Phi(\alpha), \quad \text{where } \alpha = \frac{u^2 u_{xt}}{u_x^3} - \frac{3uu_t}{u_x^2} \text{ and } \frac{\Phi^3}{(2\Phi - 2 - \alpha)^2} = \kappa.$$

In particular for $\kappa = 0$ one retrieves the heat equation and, as Bluman and Cole prove, 'the heat equation is the only polynomial partial differential equation of the second order in two independent variables invariant under the group of the heat equation'.

The construction of all differential equations which admit a given group of symmetries continued in a sequence of research papers apparently for the first time in the early eighties following the initial publication in [26]. More specifically Rosenhaus observed the one-to-one correspondence which may be exhibited between a given differential equation and its group of symmetries (invariance group); his studies revealed such a property for the Monge–Ampère equation in two dimensions which was proven to be intimately related with its point symmetries only. Rosenhaus also claimed that the equation of minimal surface, namely

$$\left(1 + u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + \left(1 + u_x^2\right)u_{yy} = 0, \tag{4}$$

is uniquely determined by its group of contact symmetries. Subsequent papers [27,28] have shown that his research was confined to partial differential equations and his findings are intriguing; we discuss more in what follows.

For the nonce no particular importance of the symmetries which comprise a complete symmetry group has been identified. However, as Rosenhaus points out [27], the simplest equations with full field-space symmetry are given by their groups of point and tangent (contact) transformations; this property distinguishes these systems from other nonlinear equations and brings them nearer to the simplest linear ones.

Another concept, apparently inspired independently, is that of Lie remarkable equations. An equation is termed to be Lie remarkable [21,22] (later revised, extended and divided into weakly and strongly Lie remarkable [18]) if it is the 'unique' differential equation to be characterised by the specific set of point symmetries only. In the subsequent terminology [18,19] weak Lie remarkability describes those differential equations which are the sole equations admitting a given set of Lie point symmetries whereas strong Lie remarkability is the property of the exact specification of the given differential equation in the whole jet space. Indeed, since the occurrence of such differential equations is quite limited, remarkable is an appropriate word accurately to delineate the above feature. Besides few differential equations possess a sufficient number of Lie point symmetries in order for a possibility to exist to be characterised by them. Note that as in the Conjecture introduced below, the jet space chosen is of fundamental importance.

In this paper we use the notion of a complete symmetry group (and Lie remarkability) to define families of equations with potentially unusual properties. The procedure we follow is that of the natural inverse problem, first applied by Bluman and Cole—the direct problem being that of determining the symmetries of a given differential equation. The probable effectiveness of this notion in unifying existing well-known equations is examined in the subsequent section. In fact it is achieved in the case of the Monge–Ampère and Bateman equations. Furthermore by using the seven-dimensional Lie group of the Born–Infeld equation we recover a highly nonlinear partial differential equation which, for specific parameter values, gives rise to equations with an unexpected abundance in Lie point symmetries and therefore physically interesting similarity solutions.

We perform a classification of differential equations according to their symmetry properties, particularly of those equations which are completely characterised by a specific algebra. A similar classification, but through a different approach, was reported, for instance, in [12]. However, our objectives are different: we begin with a differential equation and aim at finding wider families of equations which would incorporate the given equation. We extend this idea by considering the joint subalgebras of two differential equations in order to arrive at a single general class which includes them both. If these joint subalgebras consist of the requisite number of symmetries, then we achieve to unify in a sense different well-known partial differential equations. To this end we make the following Conjecture

Conjecture: The minimum number of symmetries necessary for the possible characterisation of a given differential equation depends upon the general initial form that we choose each time and equals the number of arguments of the arbitrary function, F, as for example in (3).

This Conjecture enables us to recover in an efficient manner the minimal-dimensional symmetry group required to specify completely a given differential equation. In the case of the heat equation, (1), it is apparent that one needs at least seven symmetries to restrict the jet-2 space in order to retrieve the equation itself. However, the six Lie point symmetries used for the heat equation [8] preclude any possibility for a complete characterisation of Eq. (1). For that reason we make good use of the infinite-dimensional subalgebra of the equation. Specifically, observe that Eq. (1) has the additional infinite-dimensional subalgebra $H_\infty = f(t, x)\partial_u$, where $f(t, x)$ is any solution of (1), reflecting its linearity. If we include the symmetry $H_7 = \partial_u$, which is a member of the aforementioned subalgebra, we succeed in recovering Eq. (1) and therefore this set characterises the heat equation completely. The requirement of closure is achieved when the whole infinite-dimensional algebra is considered. In fact we obtain the same result for any $f(t, x)$ chosen. This motivates us to include for the first time in the literature infinite-dimensional subalgebras for the complete characterisation and classification of differential equations.

In Section 2 we investigate the differential equations which are in one-to-one correspondence with the Lie groups of the Monge–Ampère equation, the Bateman equation and the Born–Infeld equation. Using the Conjecture described above we find all minimal-dimensional subalgebras of these groups that characterise the three aforementioned equations. Note that in the case of the Born–Infeld equation the seven Lie point symmetries, although seven, are not sufficient to characterise fully the equation. Nevertheless the Lie group of this equation is in one-to-one correspondence with a new, very complicated and highly nonlinear one-parameter family of partial differential equations. In Section 3 we start from the symmetries of the Monge–Ampère and Bateman equations and construct once again a new interesting and quite simple class of partial differential equations which gives rise to these two equations as subcases. In Section 4 we present an implementation in *Mathematica* which enables one to tackle computationally tedious calculations. We report our results in the last section and conclude with some interesting directions for future research.

2. Three well-known differential equations

Before we present the three equations we consider the Burgers equation

$$u_t + uu_x - au_{xx} = 0, \quad \text{where } a \neq 0, \tag{5}$$

in order to illustrate a point.

Eq. (5) possesses five Lie point symmetries, *videlicet*

$$\begin{aligned} &\Gamma_1 = \partial_t \qquad \Gamma_4 = 2t\partial_t + x\partial_x - u\partial_u \\ &\Gamma_2 = \partial_x \qquad \Gamma_5 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u. \\ &\Gamma_3 = t\partial_x + \partial_u. \end{aligned} \tag{6}$$

The algebra of these Lie point symmetries is $sl(2, R) \oplus_s 2A_1$. The action of all symmetries in (6) on $u_t = F(t, x, u, u_x, u_{xx})$ gives (5). Of course this is not an instance of a complete symmetry group since the general second-order partial differential equation in two dimensions has the form $u_{xx} = F(t, x, u, u_t, u_x, u_{tt}, u_{tx})$. However, if one considers the class of $1 + 1$ evolution equations, then (6) is the complete symmetry group of the Burgers equation, (5). Note that even this attribute is not mentioned without reason; Rosenhaus considered the Sine–Gordon and the Korteweg–de Vries equations,

$$u_{xt} = \sin u \quad \text{and} \quad u_t + auu_x + u_{xxx} = 0, \quad a \text{ a constant},$$

respectively, to show the fate of partial differential equations without field-space symmetry. In fact their (point and contact) symmetry groups do not even suffice for them to be completely characterised in the submanifold of the jet-2 space as is the case for (5).

2.1. The Monge–Ampère equation

The Monge–Ampère equation exhibits many forms and is thoroughly investigated in [26–29,22,18,19]. We briefly present certain results for the sake of the discussions below. The specific Monge–Ampère equation which we study is of the second order in two independent variables, namely

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{7}$$

and admits the fifteen Lie point symmetries [13]

$$\begin{array}{llll} \Delta_1 = \partial_x & \Delta_2 = \partial_y & \Delta_3 = \partial_u & \Delta_4 = x\partial_x \\ \Delta_5 = y\partial_y & \Delta_6 = u\partial_u & \Delta_7 = u\partial_y & \Delta_8 = x\partial_y \\ \Delta_9 = xu\partial_x + yu\partial_y + u^2\partial_u & \Delta_{10} = y\partial_x & \Delta_{11} = y\partial_u & \\ \Delta_{12} = x^2\partial_x + xy\partial_y + xu\partial_u & \Delta_{13} = x\partial_u & \Delta_{14} = u\partial_x & \\ \Delta_{15} = xy\partial_x + y^2\partial_y + yu\partial_u. & & & \end{array} \tag{8}$$

Note that Eq. (7) is a special case (constant equals zero, *i.e.* of zero Gaussian curvature) of the more general Monge–Ampère equation of constant (K) Gaussian curvature

$$u_{xx}u_{yy} - u_{xy}^2 = K\left(1 + u_x^2 + u_y^2\right). \tag{9}$$

The symmetries in (8) constitute a representation of the fifteen-dimensional algebra $sl(4, R)$. Eq. (7) was examined in [26,27] and was found to be uniquely determined by this group; a property later named Lie remarkability [21,22] and subsequently turned to weak Lie remarkability [18,19].

Of course not all elements of the algebra are necessary for the purpose. All seven-dimensional subalgebras of $\{\Delta_1 - \Delta_8, \Delta_{11}, \Delta_{13}\}$ that fully characterise Eq. (7) are[5]

$$\begin{array}{lllllll} \Delta_1, & \Delta_2, & \Delta_3, & \Delta_4, & \Delta_8, & \Delta_{11}, & \Delta_{13} \\ \Delta_2, & \Delta_3, & \Delta_5, & \Delta_6, & \Delta_8, & \Delta_{11}, & \Delta_{13} \\ \Delta_1, & \Delta_2, & \Delta_3, & \Delta_4, & \Delta_5, & \Delta_8, & \Delta_{13} \\ \Delta_1, & \Delta_2, & \Delta_3, & \Delta_4, & \Delta_6, & \Delta_8, & \Delta_{13} \\ \Delta_1, & \Delta_2, & \Delta_3, & \Delta_4, & \Delta_7, & \Delta_8, & \Delta_{13} \\ \Delta_2, & \Delta_3, & \Delta_5, & \Delta_6, & \Delta_7, & \Delta_8, & \Delta_{13}. \end{array} \tag{10}$$

The 'common' subalgebra of (10) is $\{\Delta_2, \Delta_3, \Delta_8, \Delta_{13}\}$, which is abelian and acts miraculously for the Monge–Ampère equation restricting the jet-2 space to

$$u_{xx} = f\left(x,\ u_y,\ u_{yy}\right) + \frac{u_{xy}^2}{u_{yy}}. \tag{11}$$

The action of the additional three symmetries in any of the cases in (10) results in $f = 0$. (Note that they all form the same algebra, $A_1 \oplus 2A_1$.) We conjecture that the optimal set [23] should in fact indicate in which instances a subalgebra of (8) results in the Monge–Ampère equation. This idea will be further explored in a separate article.

2.2. The Bateman equation

The two-dimensional Bateman equation [7] is

$$u_{xx}u_y^2 + u_{yy}u_x^2 - 2u_xu_yu_{xy} = 0 \tag{12}$$

and possesses the symmetries

$$\begin{array}{lll} \beta_1 = \phi_1(u)\partial_y & \beta_2 = x\phi_2(u)\partial_y & \beta_3 = xy\phi_3(u)\partial_x + y^2\phi_3(u)\partial_y \\ \beta_4 = \phi_4(u)\partial_u & \beta_5 = y\phi_5(u)\partial_y & \beta_6 = x^2\phi_6(u)\partial_x + xy\phi_6(u)\partial_y \\ \beta_7 = \phi_7(u)\partial_x & \beta_8 = x\phi_8(u)\partial_x & \beta_9 = y\phi_9(u)\partial_x \end{array} \tag{13}$$

with the infinite-dimensional algebra $\{A_1 \oplus_s sl(3, R)\}_\infty$, where the innovative notation is meant to indicate the infinite number of functions $\phi_i(u)$ which can appear in the coefficient functions. When we select all the $\phi_i(u),\ i = 1, 2, 4\text{–}8$, to be unity, Eq. (12) can be shown by a series of elementary calculations to be Lie remarkable. In fact this result holds for

[5] In fact Rosenhaus [27] obtains the same result by using eight symmetries when one needs only seven!

any choice of ϕ_i. Note that the specification of a differential equation by its infinite-dimensional algebra to our knowledge has never before been reported.

We compute all seven-dimensional subalgebras of the above symmetry group, (13), in the case that all coefficient functions are set equal to one. The only interesting case is drawn from the symmetries $\{\partial_y,\ x\partial_y,\ \partial_u,\ y\partial_y,\ \partial_x,\ x\partial_x,\ y\partial_x\}$ which characterise completely the equation

$$\left(u_{xx}u_{yy}-u_{xy}^2\right)+c\left(u_{xx}u_y^2+u_x^2u_{yy}-2u_xu_yu_{xy}\right)=0,$$

that is the Monge–Ampère equation plus c times the Bateman equation! The four remaining seven-dimensional subalgebras give rise to some very complicated partial differential equations which could be viewed as 'nonlinear' instances of the Bateman equation.

Rosenhaus [27] consigns the Bateman equation, (12), to be a trivial case since under the hodograph transformation [24],

$$u(x,\ y)=X,\qquad y(X,\ Y)=Y,\qquad x(X,\ Y)=U(X,\ Y), \tag{14}$$

it is equivalent to $U_{YY}=0$ and therefore can be excluded from any consideration.

2.3. The Born–Infeld equation

Consider the Born–Infeld equation [9],

$$(1-u_t^2)u_{xx}+2u_xu_tu_{tx}-(1+u_x^2)u_{tt}=0. \tag{15}$$

Eq. (15) possesses the symmetries

$$\begin{array}{lll}\Gamma_1=\partial_t & \Gamma_4=x\partial_t+t\partial_x & \Gamma_7=t\partial_t+x\partial_x+u\partial_u\\ \Gamma_2=\partial_x & \Gamma_5=u\partial_t+t\partial_u & \\ \Gamma_3=\partial_u & \Gamma_6=x\partial_u-u\partial_x & \end{array} \tag{16}$$

with the algebra $\{so(2,\,1)\oplus A_1\}\oplus_s 3A_1$. In this case the dimension of the algebra equals the minimal dimension that is required, according to the Conjecture, for the characterisation of the $J^2(\mathbb{R}^2)$ space. This fact abates significantly the possibility that the Born–Infeld equation be Lie remarkable.

Using the symmetries Γ_1, Γ_2, Γ_3 and Γ_7 firstly and acting their suitable extensions on

$$u_{tt}=F(t,\ x,\ u,\ u_t,\ u_x,\ u_{tx},\ u_{xx})$$

we obtain

$$u_{tt}=u_{tx}G\left(u_t,\ u_x,\ v\right),\quad \text{where } v=\frac{u_{xx}}{u_{tx}}. \tag{17}$$

The second extensions of the remaining symmetries are

$$\begin{aligned}\Gamma_4^{[2]}&=x\partial_t+t\partial_x-u_x\partial_{u_t}-u_t\partial_{u_x}-2u_{tx}\partial_{u_{tt}}-(u_{xx}+u_{tt})\partial_{u_{tx}}-2u_{tx}\partial_{u_{xx}}\\ \Gamma_5^{[2]}&=u\partial_t+t\partial_u+(1-u_t^2)\partial_{u_t}-u_tu_x\partial_{u_x}-3u_tu_{tt}\partial_{u_{tt}}-(2u_tu_{tx}+u_xu_{tt})\partial_{u_{tx}}-(u_tu_{xx}+2u_xu_{tx})\partial_{u_{xx}}\\ \Gamma_6^{[2]}&=-u\partial_x+x\partial_u+u_tu_x\partial_{u_t}+(1+u_x^2)\partial_{u_x}+(2u_tu_{tx}+u_xu_{tt})\partial_{u_{tt}}+(u_tu_{xx}+2u_xu_{tx})\partial_{u_{tx}}+3u_xu_{xx}\partial_{u_{xx}}\end{aligned}$$

and when operated on (17) the result is

$$(2-v^2-vG)\frac{\partial G}{\partial v}+u_x\frac{\partial G}{\partial u_t}+u_t\frac{\partial G}{\partial u_x}=2-vG-G^2 \tag{18}$$

$$(2u_t-2u_x+u_xG-vu_t)\frac{\partial G}{\partial v}+(1-u_t^2)\frac{\partial G}{\partial u_t}-u_tu_x\frac{\partial G}{\partial u_x}=(u_xG-u_t)G \tag{19}$$

$$(v^2u_t-vu_x)\frac{\partial G}{\partial v}-(1+u_x^2)\frac{\partial G}{\partial u_x}-u_tu_x\frac{\partial G}{\partial u_t}=(vu_t+u_x)G-2u_t. \tag{20}$$

Eqs. (18)–(20) are obviously too complex for a modest mathematical treatment; we resort to computational facilities to ease the calculations involved and minimise the proneness for error (see Section 4). As system (18)–(20) may be viewed as an overdetermined system of three first-order partial differential equations involving three derivatives of the first order (for the three characteristics, v, u_t and u_x) for the single function, G, we proceed algebraically. The resulting system comprises three first-order ordinary differential equations, which are solved to give

$$u_{tt}=\frac{1}{k(1+u_x^2)^2}\left\{2ku_tu_x\left(1+u_x^2\right)u_{xt}+\left(1+u_x^2\right)(4pB+(k-8)u_{xx})-\left(4pB+\left(ku_x^2+k-8\right)\right)u_t^2u_{xx}\right\}, \tag{21}$$

where $p = \pm 1$ and

$$B^2 = \frac{1}{u_t^2 - u_x^2 - 1}\left\{2ku_t u_x u_{xx} u_{xt}\left(1+u_x^2\right) - k\left(1+u_x^2\right)^2 u_{xt}^2 + \left[(k-4)\left(1+u_x^2\right) - u_t^2\left(k-4+ku_x^2\right)\right]u_{xx}^2\right\} \tag{22}$$

with $k \neq 0$ a real parameter which characterises the above one-parameter class of partial differential equations.

It also produces the singular solution, $u_{tt}u_{xx} - u_{tx}^2 = 0$, which we recognise as the Monge–Ampère equation.

Eq. (21) has the initial seven symmetries for both values of p and all values of k, apart from the case $k = 4$ (and $p = \pm 1$) which leads to the equation

$$\begin{aligned} u_{tt} = {} & \frac{1}{(1+u_x^2)^2}\left\{\left(1+u_x^2\right)\left[2p\left(1-u_t^2+u_x^2\right)^{1/2}u_{tx} - u_{xx}\right] - u_t^2\left(-1+u_x^2\right)u_{xx}\right. \\ & \left. + \; 2u_t u_x\left[\left(1+u_x^2\right)u_{tx} - p\left(1-u_t^2+u_x^2\right)^{1/2}u_{xx}\right]\right\}, \end{aligned} \tag{23}$$

which possesses an additional three-dimensional abelian algebra spanned by the symmetries

$$\begin{aligned} E_1 &= \left(t^2+x^2+u^2\right)\partial_t + 2tx\partial_x + 2tu\partial_u \\ E_2 &= 2tu\partial_t + 2xu\partial_x + \left(t^2+u^2-x^2\right)\partial_u \\ E_3 &= 2tx\partial_t + \left(t^2+x^2-u^2\right)\partial_x + 2xu\partial_u. \end{aligned} \tag{24}$$

Using the above set of symmetries we construct a similarity solution,

$$u(x,\ t) = c_1 + a\sqrt{(t-c_2)^2 - (x-c_3)^2 + c_4}, \quad a = \pm 1,$$

which can be written in the following implicit form,

$$(u-c_1)^2 + (x-c_3)^2 - (t-c_2)^2 = c_4. \tag{25}$$

Eq. (25) defines a surface in $\mathbb{R}^3$; for $c_4 > 0$ the surface is an hyperboloid of one sheet, for $c_4 = 0$ it is a cone whereas for $c_4 < 0$ it is an hyperboloid of two sheets. Although Eq. (23) possesses ten symmetries and the surfaces, (25), are both Lorenz invariant and invariant under the translations of $\mathbb{R}^3$, it is not invariant under all the rotations of $\mathbb{R}^3$ (actually only around the t-axis). This fact indicates that the ten-parameter algebra is not a representation of the Poincaré group.

Most importantly though Eq. (21) drops to the Born–Infeld equation, (15), for $k \to \infty$. The Born–Infeld equation appears not to be Lie remarkable despite all the Lie point symmetries it possesses. Of course it is deduced as an asymptotic case from the equation invariant under the seven-dimensional algebra, (21). Rosenhaus [27] proves that the Born–Infeld equation[6] is in one-to-one correspondence with its group of contact symmetries. To be more precise the correspondence is one-to-one if we exclude the Monge–Ampère equation, (7).

3. An interesting class of equations

Consider the partial differential equation

$$\left(u_{xx}u_{yy} - u_{xy}^2\right)u_y^2 = c\left(u_y u_{xy} - u_x u_{yy}\right)^2, \tag{26}$$

which, as far as we know, has never before been reported.

In passing we note that (26) can be equally written as

$$u_{xx}u_{yy} - u_{xy}^2 = cu_y^2\left[\left(\frac{u_x}{u_y}\right)_{,y}\right]^2. \tag{27}$$

Eq. (26) possesses the eight-dimensional algebra of Lie point symmetries Δ_1–Δ_8 with the algebra $\{A_2 \oplus A_2\} \oplus_s \{A_1 \oplus_s 3A_1\}$. The complete symmetry group of (26) comprises all those Lie point symmetries and in that sense Eq. (26) is said to be Lie remarkable.

Again not all eight symmetries are necessary for the purpose; every seven-dimensional subalgebra of Δ_1–Δ_8 with each missing one symmetry in turn, except of the omission of $\Delta_2 = \partial_y$, is appropriate. This means that ∂_y has an extra meaning for this equation.

It is noteworthy to remark the cases $c = -1$ and $c = 0$. For $c = 0$ Eq. (26) reduces to a second-order Monge–Ampère equation in two independent variables. For $c = -1$ Eq. (26) gives rise to the two-dimensional Bateman equation. In fact

[6] Note that the Born–Infeld equation, (15), is actually the pseudoeuclidean analogue of the equation of minimal surface, (4), *i.e.* the former is transformed to the latter by a complex change of time, t.

this is exactly how we were able to retrieve Eq. (26). We observe that the 'common' subalgebra of the Monge–Ampère and Bateman equations is

$$\{\partial_x,\ \partial_y,\ \partial_u,\ x\partial_x,\ y\partial_y,\ u\partial_u,\ u\partial_y,\ x\partial_y,\ y\partial_x,\ u\partial_x\}. \tag{28}$$

This common subalgebra could in theory lead to an equation comprising at least the two original equations. That was not the case. However, the eight-dimensional subalgebra of (28), Δ_1–Δ_8, fulfilled our expectations.

An impressive by-result is the following: All Δ_1–Δ_8 symmetries plus Δ_{14} comprise the complete symmetry group of just Eq. (12). However, if we exclude Δ_7 from the set, the remaining symmetries characterise both (7) and (12) completely! This appears to our knowledge to be the first instance that a group of symmetries completely characterises two distinct partial differential equations.

Note that the symmetries Δ_1–Δ_5, Δ_8, Δ_{10} result in

$$u_{xx}u_y^2 + cu_{xx}u_{yy} + u_{yy}u_x^2 - u_{xy}(2u_xu_y + cu_{xy}) = 0. \tag{29}$$

The Lie point symmetries of (29) are

$$\begin{array}{lll}
\Sigma_1 = \partial_x & \Sigma_2 = \partial_y & \Sigma_3 = \partial_u \\
\Sigma_4 = x\partial_x & \Sigma_5 = y\partial_x & \Sigma_6 = \exp[u/c]\partial_x \\
\Sigma_7 = x\partial_y & \Sigma_8 = y\partial_y & \Sigma_9 = \exp[u/c]\partial_y \\
\Sigma_{10} = x\exp[-u/c]\partial_u & \Sigma_{11} = y\exp[-u/c]\partial_u & \Sigma_{12} = \exp[-u/c]\partial_u \\
\Sigma_{13} = x\left(x\partial_x + y\partial_y + c\partial_u\right) & & \\
\Sigma_{14} = y\left(x\partial_x + y\partial_y + c\partial_u\right) & & \\
\Sigma_{15} = \exp[u/c]\left(x\partial_x + y\partial_y + c\partial_u\right). & &
\end{array} \tag{30}$$

The number of Lie point symmetries of (29) is the same as that for the Monge–Ampère equation, (7). In fact the transformation $u = c\log\, w$ transforms (29) to (7).

4. Computer algebraic tools

As the complexity of the equations under investigation increased so did the amount of the calculations involved. Hence the need for a computer algebra system for the tasks at hand became apparent. We chose the package *SYM* [10,11] of the program *Mathematica*. The package, written in the powerful native symbolic language of *Mathematica*, provides over seventy functions for the symmetry analysis of systems of differential equations and the algebraic manipulation of Lie algebras among other things. Furthermore its modular framework enables one to create readily new commands suitably adjusted to one's needs. Additionally the package defines a more suitable environment for problems involving differential equations and algebraic objects making easier the handling of the intermediate and final results. Last but not the least we would like to mention that among its features are its reliability (it has validating facilities), its speed and low memory consumption.

For our purposes, apart from the standard commands provided from the package for specifying the symmetries of a differential equation, we have constructed an ensemble of functions for the characterisation of a differential equation using a given set of symmetries. These are

CharacteristicDEq: Function for obtaining the differential equation(s) characterised by a given set of symmetries. Its arguments are the jet space in which we seek the differential equation given as an explicit function, e.g. $u_{tt} = F(x, t, u(x, t), u_x, u_t, u_{xt}, u_{xx})$ for the space $J^2(\mathbb{R}^2)$ and a set of symmetries. The output of this function is a differential equation characterised by the given set of symmetries. For example the command `CharacteristicDEq[D[u[x,t],t] == F[x,t,u[x,t],D[u[x,t],x],D[u[x,t],x,x]], Partial[x]]` gives the equation $u_t = F[t, u, u_x, u_{xx}]$.

CompleteSymmetryAlgebraQ: This is a query function (gives true or false as the output). Its arguments are a differential equation and a set of symmetries. It gives `True` if the set of symmetries characterises the given equation and `False` otherwise.

CompleteSymmetries: This is the most automatic function of the three. Its argument is just a differential equation. It finds its Lie point symmetries, determines all the possible subalgebras and returns as the output the smallest subalgebra that characterises the differential equation. If the equation cannot be characterised by the algebra or a subalgebra that it possesses then it returns the empty set {}.

The above three commands are included in the latest version of the package.

5. Concluding remarks

Given a differential equation one is primarily interested in solving it by any means. Before the audacity of Lie's thinking people devised *ad hoc* methods which would give results only to a restricted set or types of equations. The introduction of symmetry in the novel way of connecting differential equations to continuous groups of transformations enabled Lie to construct a concrete theory to achieve the possible solvability or integrability of differential equations. Therefore symmetries are intimately connected to the direct (or natural) problem of being the output rather than the input; given a differential equation there is a methodology to compute all symmetries that the former possesses. However, there is an equally natural inverse problem which identifies all differential equations (according to the desired form) invariant under a given symmetry group. It is this approach that we tried to highlight in this paper.

Our starting point was a set of three partial differential equations with known general solutions, particular solutions, first integrals *etc* [25]. From their outset the Monge–Ampère equation (studied by G Monge in 1784 and later by AM Ampère in 1820), the Bateman equation [7] and the Born–Infeld equation [9] have been studied in great detail. These equations are rich in Lie point symmetries and this good fortune was not a shot in the dark; they were chosen as such exceptional examples in order to provide opulent symmetry groups. In such a way we were able to put the inverse problem to good use and construct interesting generalisations which had the aforementioned equations as special and even asymptotic cases. These new classes of equations reported in this article are (26), (21) with (22) and (23).

The analytic procedure reported in this paper is in fact known already for some decades if not as far back as the times of Lie when he was able to determine the general second-order ordinary differential equation that admitted a two-dimensional (Lie, as was later called!) algebra. In this paper we have also implemented the package *SYM* to ease the calculations considerably and produce results that are not of questionable validity. When the problem attacked becomes too complicated even for computational treatment, an interactive study with the commands is required.

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References

[1] K. Andriopoulos, P.G.L. Leach, G.P. Flessas, Complete symmetry groups of ordinary differential equations and their integrals: Some basic considerations, Journal of Mathematical Analysis and Applications 262 (2001) 256–273.
[2] K. Andriopoulos, P.G.L. Leach, The economy of complete symmetry groups for linear higher-dimensional systems, Journal of Nonlinear Mathematical Physics 9 (s-2) (2002) 10–23.
[3] K. Andriopoulos, P.G.L. Leach, M.C. Nucci, The Ladder problem: Painlevé integrability and explicit solution, Journal of Physics A: Mathematical and General 36 (2003) 11257–11265.
[4] K. Andriopoulos, P.G.L. Leach, The complete symmetry group of the generalised hyperladder problem, Journal of Mathematical Analysis and Applications 293 (2004) 633–644.
[5] K. Andriopoulos, Complete symmetry groups and Lie remarkability, in: G. Gaeta, R. Vitolo, S. Walcher (Eds.), Symmetry and Perturbation Theory – SPT 2007, World Scientific, Singapore, 2007, pp. 237–238.
[6] K. Andriopoulos, P.G.L. Leach, A. Maharaj, On Differential Sequences, 2008, arXivid:0704.3243.
[7] H. Bateman, Notes on a differential equation which occurs in the two-dimensional motion of a compressible fluid and the associated variational problems, Proceedings of the Royal Society A 125 (1929) 598–618.
[8] G.W. Bluman, J.D. Cole, Similarity Methods for Differential Equations, Springer, New York, 1974.
[9] M. Born, L. Infeld, Foundations of the new field theory, Proceedings of the Royal Society of London A 144 (1934) 425–451.
[10] S. Dimas, D. Tsoubelis, SYM: A new symmetry-finding package for Mathematica, in: N.H. Ibragimov, C. Sophocleous & P.A. Damianou (Eds.), Group Analysis of Differential Equations, University of Cyprus, Nicosia, 2005, pp. 64–70.
[11] S. Dimas, D. Tsoubelis, A new Mathematica-based program for solving overdetermined systems of PDEs, in: 8th International Mathematica Symposium, Avignon, France, 2006.
[12] E. Hillgarter, A contribution to the symmetry classification problem for second-order PDEs in $z(x, y)$, IMA Journal of Applied Mathematics 71 (2006) 210–231.
[13] N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 1: Symmetries, Exact Solutions and Conservation Laws, CRC Press, Boca Raton, FL, 1994.
[14] J. Krause, On the complete symmetry group of the classical Kepler system, Journal of Mathematical Physics 35 (1994) 5734–5748.
[15] J. Krause, On the complete symmetry group of the kepler problem, in: A. Arima (Ed.), Proceedings of the XXth International Colloquium on Group Theoretical Methods in Physics, World Scientific, Singapore, 1995, pp. 286–290.
[16] P.G.L. Leach, M.C. Nucci, S. Cotsakis, Symmetry, singularities and integrability in complex dynamics V: Complete symmetry groups of nonintegrable ordinary differential equations, Journal of Nonlinear Mathematical Physics 8 (2001) 475–490.
[17] P.G.L. Leach, K. Andriopoulos, M.C. Nucci, The Ermanno–Bernoulli constants and representations of the complete symmetry group of the Kepler Problem, Journal of Mathematical Physics 44 (2003) 4090–4106.
[18] G. Manno, F. Oliveri, R. Vitolo, On differential equations characterized by their Lie point symmetries, Journal of Mathematical Analysis and Applications 332 (2007) 767–786.
[19] G. Manno, F. Oliveri, R. Vitolo, Differential equations uniquely determined by algebras of point symmetries, Theoretical and Mathematical Physics 151 (2007) 843–850.
[20] S. Myeni, P.G.L. Leach, Nonlocal symmetries and the complete symmetry group of $1 + 1$ evolution equations, Journal of Nonlinear Mathematical Physics 13 (2006) 377–392.
[21] F. Oliveri, Lie symmetries of differential equations: Direct and inverse problems, Note di Matematica 23 (2004/2005) 195–216.
[22] F. Oliveri, Sur une propriété remarquable des équations de Monge–Ampère, Rendiconti del Circolo Matematico di Palermo Serie II 78 (2006) 243–257.

[23] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.
[24] N. Petersson, Hodograph Transformations and Second Order Nonlinear Evolution Equations, Dissertation, Luleå University of Technology, Luleå, Sweden, 2002.
[25] A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC, 2004.
[26] V. Rosenhaus, On one-to-one correspondence between the equation and its group. The Monge–Ampère equation, Preprint, F-18 Acad Sci Estonian SSR – Tartu, 1982.
[27] V. Rosenhaus, The unique determination of the equation by its invariance group and field-space symmetry, Algebras, Groups and Geometries 3 (1986) 148–166.
[28] V. Rosenhaus, Groups of invariance and solutions of equations determined by them, Algebras, Groups and Geometries 5 (1988) 137–150.
[29] C. Udrişte, N. Bîlă, Symmetry Lie group of the Monge–Ampère equation, Balkan Journal of Geometry and its Applications 3 (1998) 121–134.